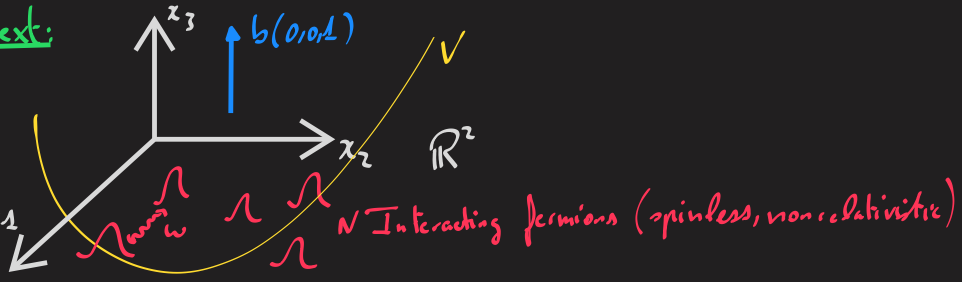
Semi-classical limit of the 2D Hartree equation in a large magnetic field

with Nicolas Rougerie

Context:



Question: dynamic when $\begin{cases} N, b \to +\infty \\ \hbar \to 0 \end{cases}$

Goal: Hartree $\to$ drifting equation

Motivation: QHE

I Model:

Magnetic Laplacian: $\mathcal{L}_b := (i\hbar\nabla + bA)^2 =: \sum'_{n\in\mathbb{N}} 2\hbar b\left(n+\frac{1}{2}\right)\Pi_n$ — n^{th} Landau level projection

Symmetric gauge: $A := \frac{X^\perp}{2} = \frac{1}{2}\binom{-x_2}{x_1}$, $\nabla\wedge A = (0,0,1)$, magnetic length: $\ell_b := \sqrt{\frac{\hbar}{b}}$,

Free ground state: $\varphi_{00} = \frac{1}{\sqrt{2\pi}\,\ell_b} e^{-\frac{|x|^2}{4\ell_b^2}}$

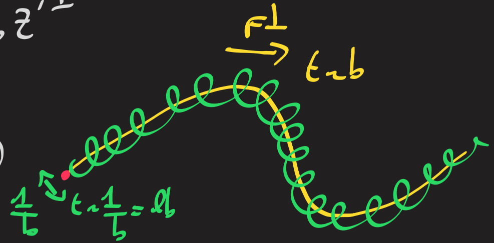
Semi-classical/Large magnetic field limit: $\ell_b \to 0$, $O(1)$ gap: $\hbar b = O(1)$

(for example $b \to +\infty$, $\hbar := \frac{1}{b} \xrightarrow[b\to+\infty]{} 0$, $\ell_b = \frac{1}{b} \xrightarrow[b\to+\infty]{} 0$)

Hartree equation: $i\hbar\partial_t\gamma = [\mathcal{L}_b + V + w*\rho_\gamma, \gamma]$ where $\gamma \in \mathcal{L}^1(L^2(\mathbb{R}^2))$, $\mathrm{Tr}(\gamma) = 1$, $\gamma \geq 0$ and $\rho_\gamma(z) := \gamma(z,z)$

Newton's dynamics: in a constant homogeneous force field F: $z'' = F + b z'^\perp$

We have $z(t) = \underbrace{\frac{|z_c'(0)|}{b}\binom{\cos(bt)}{\sin(bt)}}_{:= z_c \text{ cyclotron}} + \underbrace{\frac{F^\perp t}{b}}_{:= z_d \text{ drift}}$ imposing $\begin{cases} z_d(0,0) = (0,0) \\ z_c(0) = \frac{|z_c'(0)|}{b}(1,0) \end{cases}$



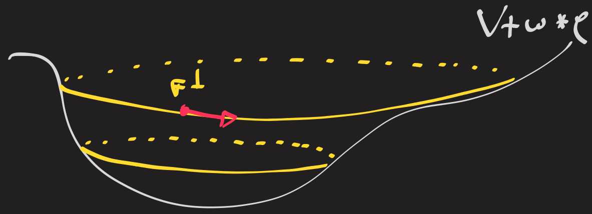
$\gamma_b(t) := \gamma(bt)$ then $i\ell_b^2\partial_t\gamma_b = [\mathcal{L}_b + V + w*\rho_{\gamma_b}, \gamma_b]$ (H)

Pauli principle: assume $\gamma_b \leq 2\pi\ell_b^2$ (FDM) Degeneracy per area in a Landau level: $\frac{1}{2\pi\ell_b^2}$

take $\gamma_b := \frac{1}{N}\sum_{i=1}^{N} |u_i\rangle\langle u_i|$ with $(u_i)_i \perp$ then $\underbrace{N := \frac{1}{2\pi\ell_b^2}}_{\text{volume } O(1)} \Rightarrow \mathrm{Tr}(\gamma_b) = 1$, $0 \leq \gamma_b \leq \frac{1}{N} = 2\pi\ell_b^2$

Drifting equation: $\partial_t\rho + \nabla^\perp(V + w*\rho)\cdot\nabla\rho = 0$ with $\rho : \mathbb{R}^2 \times \mathbb{R}_+ \to \mathbb{R}_+$

$F = \nabla(V + w*\rho)$
(charge $= -1$)



Results:

theorem:

Let γ_b solve (H) with $\gamma_b(0)$ a FDM, ρ solves (D) and $p > 7$. Assume:

- $\mathrm{Tr}\left(\gamma_b(0)\left(\mathcal{L}_b + V + \frac{1}{2}w*\rho_{\gamma_b(0)}\right)\right) < C$
- $\mathrm{Tr}(\gamma_b(0)|X|^p) < C$
- $V, w \in W^{4,\infty}(\mathbb{R}^2)$

then $\forall \varphi \in W^{1,\infty} \cap H^1(\mathbb{R}^2)$, $\forall t \in \mathbb{R}_+$,

$$\left|\int \varphi(\rho_{\gamma_b}(t) - \rho(t))\right| \leq \|\nabla\varphi\|_{L^\infty} e^{C_1(V,w)t}\left(W_1(\rho_{\gamma_b}(0), \rho(0)) + C_2(V,w)\,\ell_b^{\min\left(2\cdot\frac{p-7}{4p-7}, \frac{2}{7}\right)}\right) + C_3(V,w)\left(\|\varphi\|_{L^\infty} + \|\nabla\varphi\|_{L^2}\right)\ell_b^{\min\left(\frac{2p-7}{4p-7}, \frac{3}{7}\right)}$$

$$\begin{tikzcd}[column sep=huge, row sep=large]
\text{Many Body Schrödinger} \arrow[r, "{N\to+\infty,\ \text{Mean field}}", "\text{Benedikter Porta Schlein 14}"'] \arrow[dr, "\text{Chen Lee Liew 21}"] \arrow[drr, bend right, "\text{Porat 22}"'] & \text{Hartree} \atop \text{Hartree/Fock} \arrow[r, "{b\to+\infty,\ \text{large magnetic field}}"] \arrow[d, "{\hbar\to0 \text{ semi-classical}}"', "\text{Lafleche Saffirio 20}"] & \text{Drifting} \\
 & \text{Vlasov} \arrow[r, "{b\to+\infty,\ \text{large magnetic field}}", "\text{Golse Saint-Raymond 99}"'] & \text{Euler Vorticity}
\end{tikzcd}$$

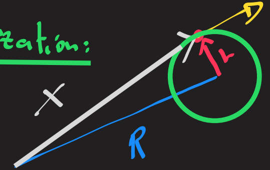
↗ $\mathbb{R}^2\times\mathbb{N}$ phase space

$\int$ 2d Coulomb interaction

. 2d Coulomb, $\hbar b \ll O(\text{interactions}) \Rightarrow$ $\mathbb{R}^2\times\mathbb{R}^2$ phase space

Quantization:

$\Pi_b := i\hbar\nabla + bA$



$r := \frac{\Pi_b^\perp}{b}$ $\quad$ $R := X - r$

$a_c^+ := \frac{r_2 + i r_1}{\sqrt{2}\ell_b}$ $\quad$ $a_d^+ := \frac{R_1 + iR_2}{\sqrt{2}\ell_b}$ independant harmonic oscillators: $\begin{cases} [a_c, a_c^+] = [a_d, a_d^+] = \mathrm{Id} \\ [a_c, a_d] = [a_c, a_d^+] = 0 \end{cases}$

$\varphi_{n,m} := \frac{(a_c^+)^n (a_d^+)^m}{\sqrt{n!\,m!}}\varphi_{0,0}$ is a basis of $L^2(\mathbb{R}^2)$ and $\Pi_n = \sum_{m\in\mathbb{N}} |\varphi_{n,m}\rangle\langle\varphi_{n,m}|$

Coherent state: Let $z\in\mathbb{C}$, $\psi_{z,n} := e^{-\frac{|z|^2}{4\ell_b^2} + \frac{\bar z}{\sqrt{2}\ell_b}a_d^+}\varphi_{n,0}$

localized at z on a length scale $\sqrt{n}\,\ell_b$ in the n^{th} landau level

$\Pi_z := \sum_{n\in\mathbb{N}} \underbrace{|\psi_{z,n}\rangle\langle\psi_{z,n}|}_{=\Pi_{z,n}}$ has kernel $\Pi_z(x,y) = \frac{1}{2\pi\ell_b^2} e^{\frac{|x-y|^2 - 2i(x^\perp\cdot y + 2z^\perp\cdot(x-y))}{4\ell_b^2}}$

so $\nabla_z^\perp \Pi_z = \frac{1}{i\ell_b^2}[\Pi_z, X]$ $\quad (*)$

Semi classical limit:

Hushimi measure: $m_{\gamma_b}(z,n) := \frac{1}{2\pi\ell_b^2}\langle\psi_{z,n}|\gamma_b\psi_{z,n}\rangle$

semi-classical density: $\rho^{x}_{\gamma_b}(z) := \frac{1}{2\pi\ell_b^2}\mathrm{Tr}(\gamma_b\Pi_z)$

truncated semi-classical density $\rho^{x,\le M}_{\gamma_b}(z) := \frac{1}{2\pi\ell_b^2}\sum_{n=0}^{M} m_{\gamma_b}(z,n) \simeq \rho_{\gamma_b}$ if $1 \ll M \ll \frac{1}{\ell_b^2}$

prop: dynamics of $\rho^{x,\le M}_{\gamma_b}$

Let $t\in\mathbb{R}_+$, $\gamma_b(t)$ be a FDM, $W\in W^{4,\infty}(\mathbb{R}^2)$ and assume $i\ell_b^2\partial_t\gamma_b = [\mathcal{L}_b + W, \gamma_b(t)]$, $\mathrm{Tr}(\gamma_b(t)\mathcal{L}_b) < C$

then $\exists$ a choice $1 \ll M \ll \frac{1}{\ell_b^2}$ such that $\forall\varphi\in L^1\cap W^{1,\infty}(\mathbb{R}^2)$,

$$\int \varphi\left(\partial_t\rho^{x,\le M}_{\gamma_b(t)} + \nabla^\perp W\cdot\nabla\rho^{x,\le M}_{\gamma_b(t)}\right) \xrightarrow[b\to+\infty]{} 0$$

Central computation:

• $\partial_t\rho^{x}_{\gamma_b}(z) = \frac{1}{2\pi\ell_b^2}\mathrm{Tr}(\Pi_z\partial_t\gamma_b) \underset{(H)}{=} \frac{1}{2\pi\ell_b^2}\cdot\frac{1}{i\ell_b^2}\mathrm{Tr}(\Pi_z[\mathcal{L}_b + W, \gamma_b]) \underset{\text{Tr-cyclicity}}{=} \frac{1}{2i\pi\ell_b^4}\mathrm{Tr}(\gamma_b[\Pi_z, \mathcal{L}_b + W])$

$= \frac{1}{2i\pi\ell_b^4}\mathrm{Tr}(\gamma_b[\Pi_z, W])$

• $\nabla^\perp W(z)\cdot\nabla\rho^{x}_{\gamma_b}(z) = -\nabla W(z)\cdot\frac{1}{2\pi\ell_b^2}\mathrm{Tr}(\gamma_b\nabla_z^\perp\Pi_z) \underset{(*)}{=} \frac{-1}{2i\pi\ell_b^4}\mathrm{Tr}(\gamma_b[\Pi_z, \nabla W(z)\cdot X])$

so $\partial_t\rho^{x}_{\gamma_b}(z) + \nabla^\perp W(z)\cdot\nabla\rho^{x}_{\gamma_b}(z) = \frac{1}{2i\pi\ell_b^4}\mathrm{Tr}\big(\gamma_b\underbrace{[\Pi_z, W - \nabla W(z)\cdot X]}_{\Pi_z(x,y)(W(y) - W(x) - (y-x)\cdot\nabla W(z))}\big)$

$\sum_{n\in\mathbb{N}}\Pi_{z,n}(x,y)$